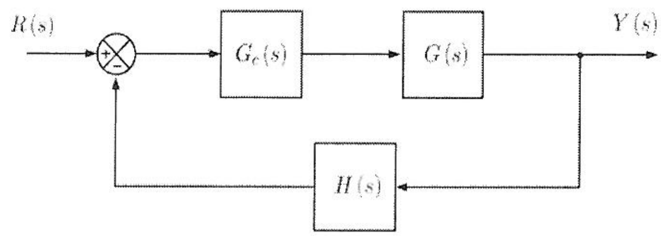
Controller Design using Asymptotic Bode Plots

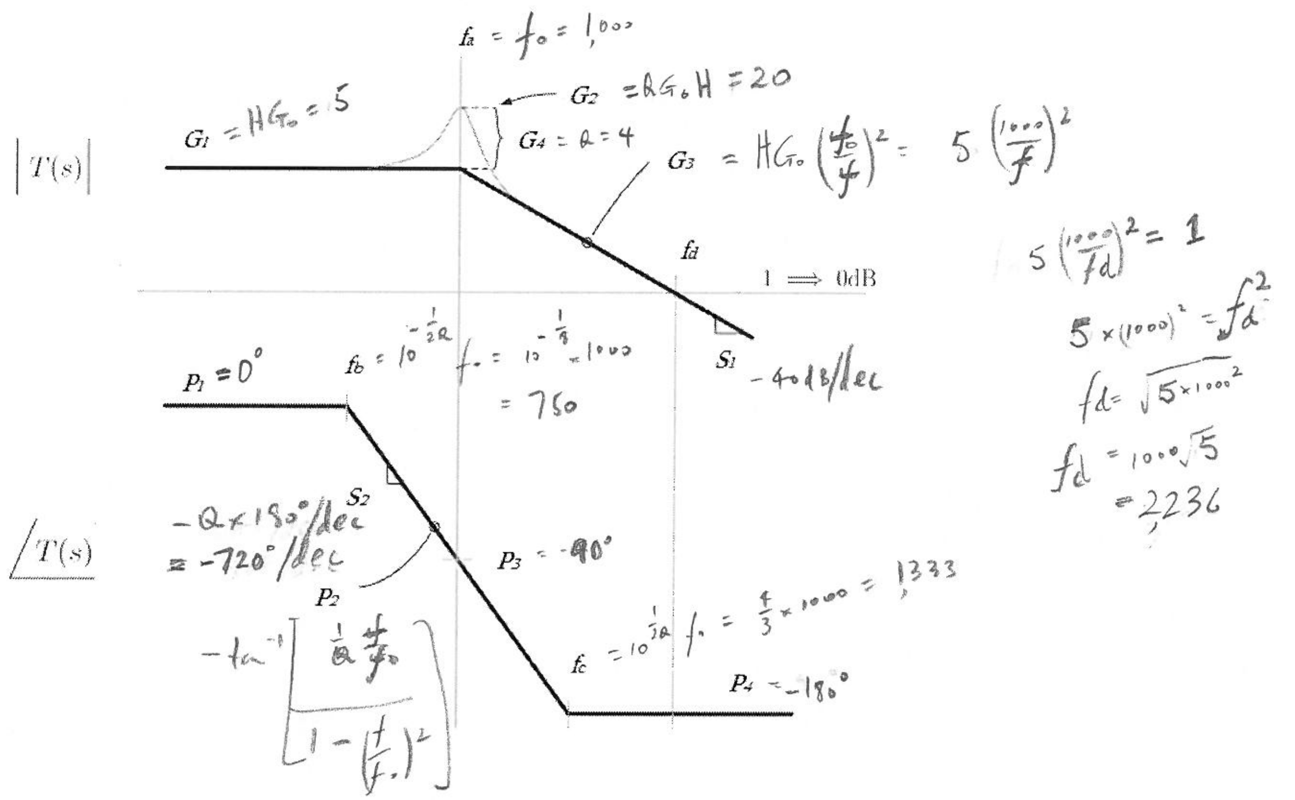
The figure below shows a closed loop control system, where the plant, $G(s)$, is given by

$$G(s) = \frac{G_o}{1 + \frac{s}{Q\omega_o} + \left(\frac{s}{\omega_o}\right)^2}$$

where $G_o = 10,\ Q = 4,\ \omega_o = 2\pi(1000)$. The feedback gain, $H(s) = {}^1/_2$. Initially the system is uncompensated, so that $G_c(s) = 1$. (Hint: $10^{-1/8} \approx \frac{3}{4}$)



The asymptotic Bode plot of the uncompensated loop gain, $T(s)\big(= G_c(s)G(s)H(s)\big)$, with $G_c(s) = 1$, is:



Determine the following:

Frequencies:

1) f_a $= f_0 = 1{,}000$
2) f_b $= 10^{-\frac{1}{2Q}} f_0 = 750$
3) f_c $= 10^{\frac{1}{2Q}} f_0 = 1333$

$f_d = \sqrt{HG_0}\, f_0 = 2{,}236$ Hz

Gains:

1) G_1 $= HG_0 = 5$
2) G_2 $= HG_0 Q = 20$
3) G_3 $= HG_0 \left(\frac{f_0}{f}\right)^2 = 5\left(\frac{1000}{f}\right)^2$
4) G_4 (this is a ratio of gains) $Q = 4$

Phase values:

1) P_1 $= 0°$
2) P_2 $= -\tan^{-1}\left[\frac{\frac{1}{Q}\frac{f}{f_0}}{1-\left(\frac{f}{f_0}\right)^2}\right]$, $Q = 4$, $f_0 = 1{,}000$
3) P_3 $= -90°$
4) P_4 $= -180°$

Gain and phase slope values:

1) S_1 -40 dB/dec
2) S_2 $-Q \times 180°/\text{dec} = -720°/\text{dec}$

Determine the phase and gain margins of this uncompenstated system:

1) Phase margin $f_d = 2{,}236$ phase $= -180°$ => $0°$ PHASE MARGIN
2) Gain margin $\phi > -180° \ \forall f$ => GM = ∞

Is the closed loop system stable? YES (PM > 0, ASYMPTOTE SAYS PM = 0° BUT ACTUAL VALUE IS SLIGHTLY > 0)

Determine the system type number. 0

Assuming a unit step input, determine the final value of the output: 1.66

Assuming an ideal gain of 2 $\left(=\frac{1}{H(0)}\right)$, determine the percentage steady state error: $\frac{2-1.66}{2} \times 100 = 17\%$

$$Y(s) = \frac{G}{1+GH} R(s) = \frac{\dfrac{G_0}{1+\frac{s}{Q\omega_0}+\left(\frac{s}{\omega_0}\right)^2}}{1+\dfrac{HG_0}{1+\frac{s}{Q\omega_0}+\left(\frac{s}{\omega_0}\right)^2}} \left(\frac{1}{s}\right)$$

FINAL VALUE THEOREM:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \frac{G}{1+GH} R(s)$$

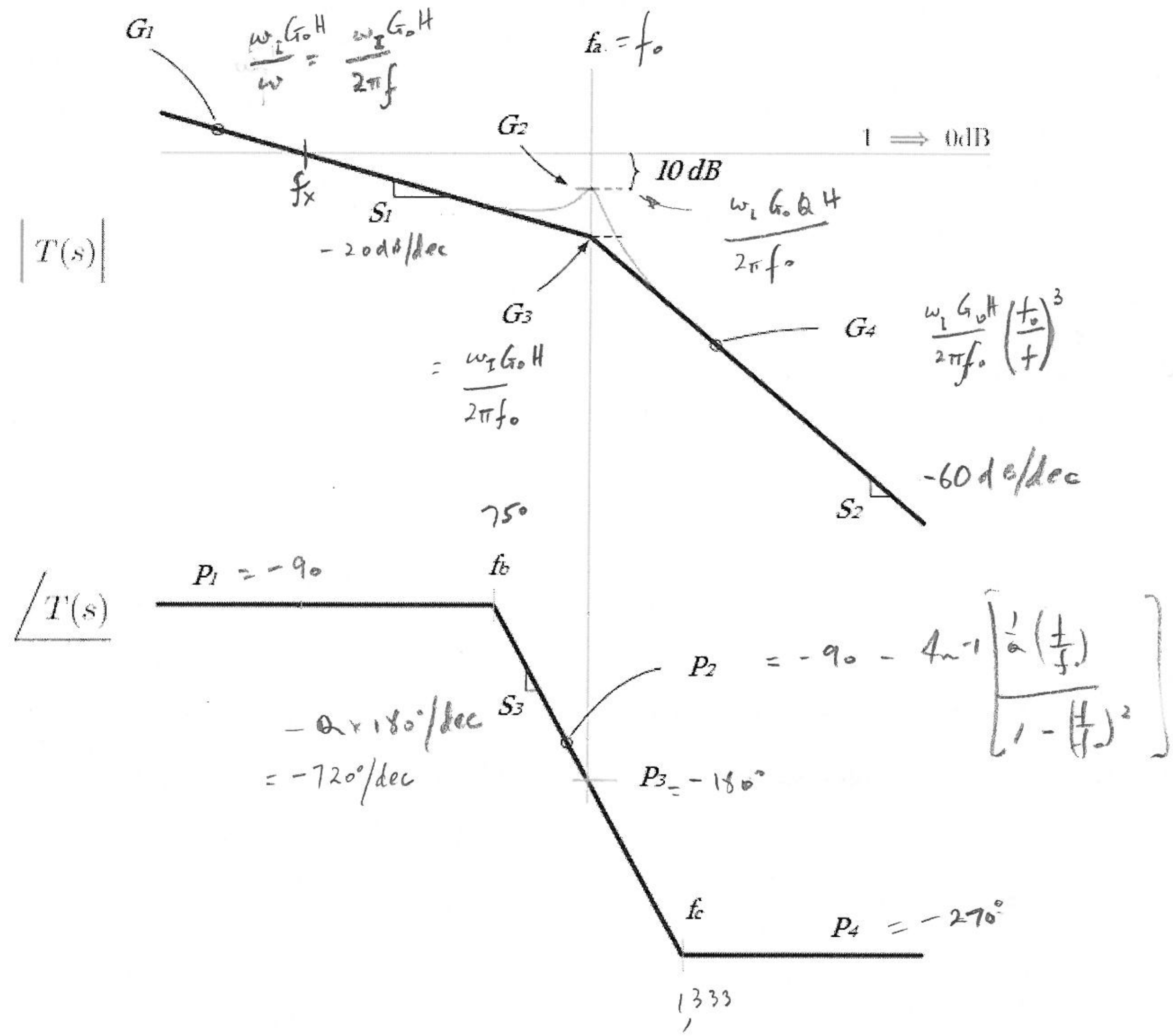
$$= \frac{G_0}{1+HG_0} = \frac{10}{1+\frac{10}{2}} = \frac{20}{12} = \frac{5}{3} = 1.66$$

Compensation:

In order to improve the performance of the system an integral compensator is considered:

$$G_c(s) = \frac{\omega_I}{s}$$

To design this compensator the parameter, ω_I, needs to be determined. This will be undertaken with the aid of the asymptotic Bode plot of the compensated loop gain, $T(s)\big(= G_c(s)G(s)H(s)\big)$, with $G_c(s) = \frac{\omega_I}{s}$, shown next:



Determine the following:

Frequencies:

1) f_a $f_o = 1{,}000$
2) f_b 750
3) f_c 1330

Gains:

1) G_1 → $\frac{\omega_I G_o H}{2\pi f}$
2) G_2 → $\frac{\omega_I G_o H Q}{2\pi f_o}$
3) G_3 $\frac{\omega_I G_o H}{2\pi f}$
4) G_4 $\frac{\omega_I G_o H}{2\pi f_o}\left(\frac{f_o}{f}\right)^3$

Phase values:

1) P_1 -90°
2) P_2 — $-90 - \tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{f}{f_o}\right)}{1-\left(\frac{f}{f_o}\right)^2}\right]$, $Q = 4$, $f_o = 1{,}000$
3) P_3 -180°
4) P_4 -270°

Gain and phase slope values:

1) S_1 $-20\,dB/dec$
2) S_2 $-60\,dB/dec$
3) S_3 $-Q \times 180^\circ/dec = -720^\circ/dec$

@ f_o $\frac{\omega_I G_o Q H}{2\pi f_o} = \frac{1}{\sqrt{10}}$ $\Rightarrow \omega_I = \frac{1}{\sqrt{10}} \frac{2\pi f_o}{G_o Q H} = 99$

Using the equations derived so far choose a value of ω_I which places the resonant Q peak at 10dB below the 0dB gain value as shown in the plot above. [Hint: -10 dB → $\frac{1}{\sqrt{10}}$]. Determine the phase and gain margins of the resulting loop gain and their associated frequencies.

1) Phase margin $PM = 180 - 90 = 90^\circ$
2) Unity gain frequency (in Hz) $= f_x$ → $\frac{\omega_I G_o H}{2\pi f_x} = 1 \Rightarrow f_x = \frac{\omega_I G_o H}{2\pi}$
3) Gain margin 10dB
4) -180° phase crossover frequency (in Hz) $f_o = 1{,}000$

$f_x = 79$ Hz.

Is the closed loop system stable? Yes (since PM > 0)

Determine the system type number. 1

Assuming a unit step input, determine the final value of the output: 2

Assuming an ideal gain of 2 $\left(= \frac{1}{H(0)}\right)$, determine the percentage steady state error: 0%

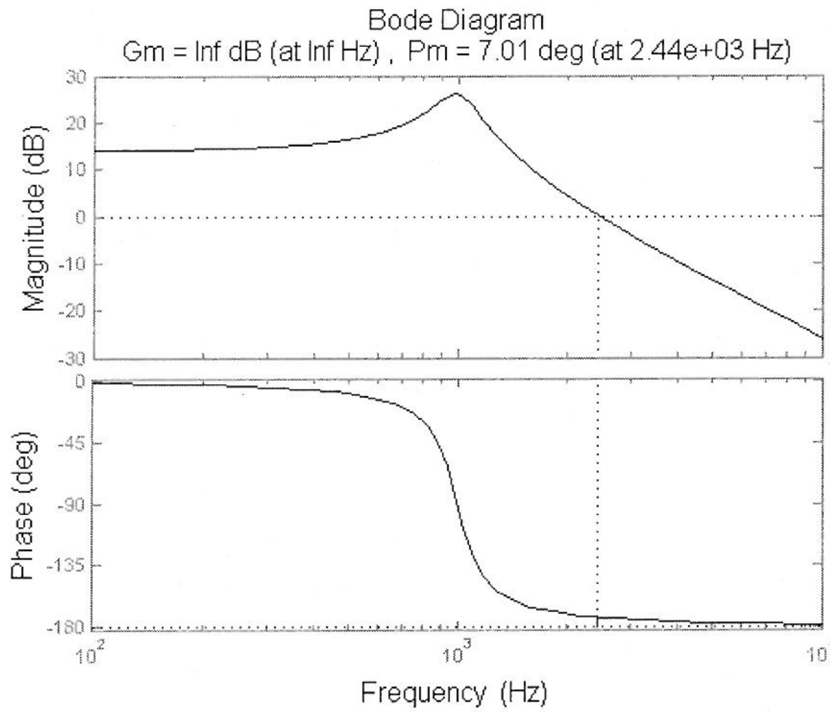
$$Y(s) = \frac{G_c G}{1 + G_c G H} R = \frac{\frac{\omega_I}{s} \frac{G_o}{1 + \frac{s}{Q\omega_o} + \left(\frac{s}{\omega_o}\right)^2}}{1 + \frac{\omega_I}{s} \frac{G_o H}{1 + \frac{s}{Q\omega_o} + \left(\frac{s}{\omega_o}\right)^2}} \left(\frac{1}{s}\right)$$

FINAL VALUE THEOREM

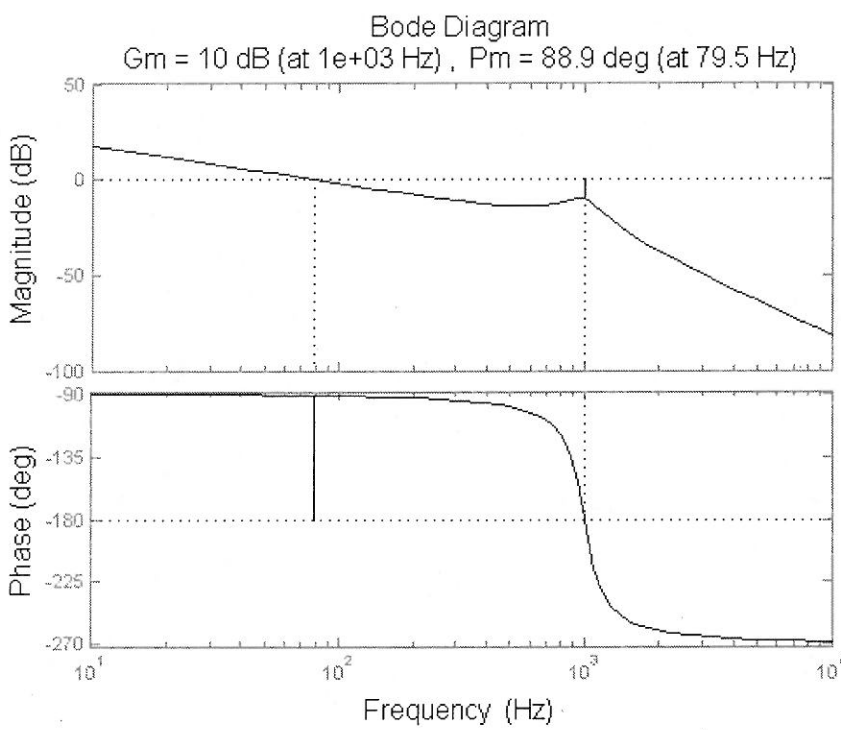
$$\Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{H} = \frac{1}{\frac{1}{2}} = 2.$$

MATLAB OBTAINED RESULTS

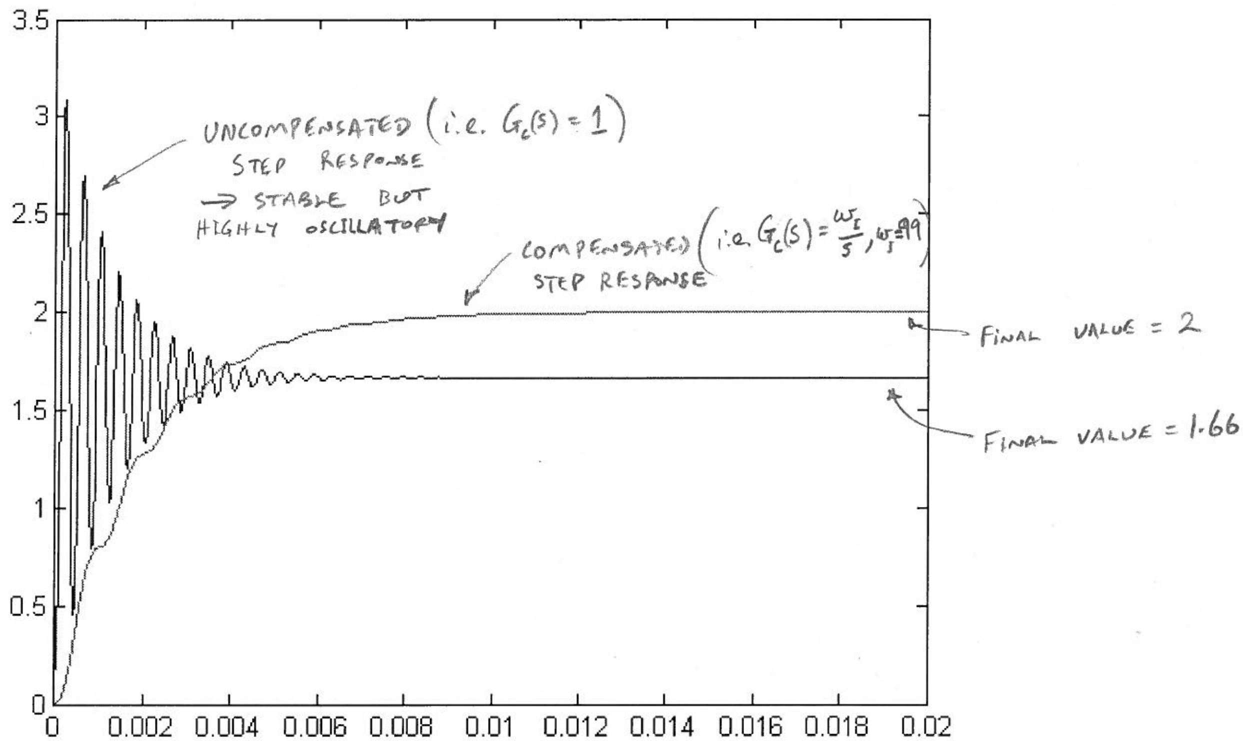
Uncompenstated loop gain:



Compenstated loop gain:



Step response:



```
% bode_prob.m
%
clear
close all
format compact

s = tf('s');

Go = 10;
fo = 1000;
wo = 2*pi*fo;
Q = 4;
w1 = 10^(-1/(2*Q))
w2 = 10^(1/(2*Q))
fL = fo*w1
fH = fo*w2


G = tf(Go,[(1/wo)^2, 1/(Q*wo), 1]);
H = 1/2;
Gc = 1;

To = Gc*G*H;

margin(To)

h = gcr;
h.AxesGrid.Xunits = 'Hz';
h.AxesGrid.TitleStyle.FontSize = 12;
h.AxesGrid.XLabelStyle.FontSize = 12;
h.AxesGrid.YLabelStyle.FontSize = 12;

Tcl = 1/H * (To/(1+To));
[yu, tu] = step(Tcl,20e-3);
%stepinfo(Tcl)
yfu = yu(end)

per_err = (1/H - yfu)/ (1/H) *100

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%   Compensated

db = 10;
gp = 10^(-db/20)

wI = gp*2*pi*fo/(Go*H*Q)

Gc = wI/s;

Tc = Gc*G*H;

figure
margin(Tc)
```

```
h = gcr;
h.AxesGrid.Xunits = 'Hz';
h.AxesGrid.TitleStyle.FontSize = 12;
h.AxesGrid.XLabelStyle.FontSize = 12;
h.AxesGrid.YLabelStyle.FontSize = 12;

Tcl = 1/H * (Tc/(1+Tc));
[yc, tc] = step(Tcl,20e-3);
%stepinfo(Tcl)
yfc = yc(end)

figure
plot(tu,yu,tc,yc)
```

%%

Results:

>> bode_prob

w1 = 0.7499

w2 = 1.3335

fL =749.8942

fH = 1.3335e+03

yfu =1.6667

per_err =16.6667

gp = 0.3162

wl = 99.3459

yfc = 1.9999